The Angel of power 2 wins

András Máthé*

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Abstract

We solve Conway's Angel Problem by showing that the Angel of power 2 has a winning strategy.

An old observation of Conway is that we may suppose without loss of generality that the Angel never jumps to a square where he could have already landed at a previous time. We turn this observation around and prove that we may suppose without loss of generality that the Devil never eats a square where the Angel could have already jumped. Then we give a simple winning strategy for the Angel.

1 Introduction

The Angel and Devil game is played on the two-dimensional lattice $\mathbb{Z}^2$, which can also be interpreted as an infinite chessboard. At the beginning the Angel is at some square, say (0,0). In his turn, the Angel can jump to any square at distance at most p from his present position (in the ℓ_∞ norm; that is, he can jump to a square which can be reached in p or less king's moves). The Devil, on his turn, eats some square. The Angel loses if he jumps to a square which has been already eaten by the Devil at some previous time, and wins if he can move forever on uneaten squares.

This game has been popularised by J.H. Conway (see e.g. [3]), and he raised the following question (the *Angel Problem*): Is it true that the Angel of sufficiently large power can defeat the Devil?

We solve this long-standing problem by showing that the Angel of power 2 can defeat the Devil (Theorem 4.6).

It is easy to show (by a simple compactness argument) that for each power p, either the Angel or the Devil has a winning strategy. It is known and easy to show that the Angel of power 1 can be trapped by the Devil. Many Angel strategies which might be expected to succeed are proved to fail (regardless of the Angel's power) [3]. Angels handicapped in different ways, like the Fool (who always increases his y coordinate), can be caught by the Devil (see [3]).

B. Bollobás and I. Leader [1] and independently M. Kutz [6] showed that *in three dimensions* the Angel of sufficiently large power can defeat the Devil. However, the technique used in three dimensions does not seem to work in two dimensions (cf. [3, Section 10]), since an Angel using this strategy can be caught by the Devil in a similar way as the Fool.

Our solution of the (two-dimensional) Angel Problem reaches back to an observation of Conway. In [3, Theorem 8.1] he showed the following: "It does no damage to the Angel's prospects if we require him after each move to eat away every square that he could have landed on at that move, but didn't." That is, if the Angel of some power can beat the Devil, then the Angel can beat the Devil such way that he never jumps to a square where he could have already landed at a previous time.

Obviously if the Angel plays this way, then there is no point in the Devil eating a square which the Angel has already used or a square where the Angel could have already jumped

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at a previous time but did not. We call the Devil who plays this way a *Nice Devil* (cf. Definitions 2.2 and 2.3).

We cannot conclude from the preceding that if the Devil can trap the Angel, then the Nice Devil can also trap the Angel; but this is what we prove in Section 2. In Section 3 we give an explicit winning strategy for the Angel of sufficiently large power against the Nice Devil, and then we conclude that the Angel of sufficiently large power can defeat the Devil (Theorem 3.3). In Section 4 we show how the Angel of power 2 can escape from the Nice Devil, and then we conclude that the Angel of power 2 can defeat the Devil (Theorem 4.6). See Section 5 for closing remarks.

Independently from the present work, three other solutions to the Angel problem have appeared also recently. B. H. Bowditch [2] has shown that the Angel of power 4 can win, P. Gács [4] has shown that the Angel of sufficiently large power can win, and O. Kloster [5] has shown that the Angel of power 2 can win.

Notation. Given two squares u and v in $\mathbb{Z}^2$, let $d(u, v)$ denote the distance of u and v in the ℓ_∞ norm. Let $B(r) = \{(x, y) : |x| \leq r, |y| \leq r\} = \{(x, y) : d((0, 0), (x, y)) \leq r\}$.

Clearly we may allow the Devil not to eat any square if he wishes on any particular turn. We may suppose that at the beginning of the game the Angel is at $(0, 0)$ and the Devil is to move.

2 The Nice Devil

We start with a trivial observation.

Claim 2.1. *If the Devil can catch the Angel of power p, then there exists a positive integer N such that the Devil can entrap the Angel in $B(N)$.*

Proof. If for every positive integer n the Angel has a strategy to make n moves without jumping on eaten squares, then a suitable limit of these strategies gives a winning strategy for the Angel to move forever without jumping on eaten squares. Hence if the Devil can catch the Angel, then there exists an n such that the Devil can catch the Angel in at most n steps. Thus, the Devil can entrap the Angel of power p in $B(np)$. □

Definition 2.2. A *Nice Devil* is a Devil who never eats a square on which the Angel has previously stayed nor a square on which the Angel could have jumped at a previous stage but did not.

Obviously the Angel can move forever while he is playing with the Nice Devil, so we modify the aim of the game.

Definition 2.3. In the game of the Angel and the Nice Devil, the Nice Devil wins if there exists a positive integer N such that he can keep the Angel trapped in the bounded domain $B(N)$. That is, the Angel wins if he can leave every bounded domain.

Theorem 2.4. *For any positive integers p and N, if the Devil can entrap the Angel of power p in $B(N)$, then the Nice Devil can also entrap the Angel of power p in $B(N)$ (that is, the Nice Devil has a strategy such that the Angel cannot leave this domain).*

Before proving the theorem precisely, we need to discuss what a strategy for the Devil is.

Definition 2.5. A *journey* of the Angel of power p is a finite sequence of squares $(v_0, v_1, \ldots, v_n)$ such that $v_0 = (0, 0)$ and $d(v_i, v_{i+1}) \leq p$ for each $0 \leq i < n$.

Hence a Devil's strategy is some function Φ which maps every (finite) journey of the Angel to some square (which is to be eaten). To be more precise, Φ maps from the journeys to $\mathbb{Z}^2 \cup \{\text{nothing}\}$, since we also allow the Devil not to eat anything if he wishes. (This way the Devil also knows which squares are already eaten on the board, so the journey of the Angel holds all the information on the game.)

We say that a journey of the Angel is *allowed* against some Devil strategy if the Angel following this journey has never jumped on an eaten square with respect to this strategy of the Devil.

Proof of Theorem 2.4. Fix a strategy Φ for the Devil with which he can entrap the Angel of power p in $B(N)$ (that is, the Angel cannot leave this domain without jumping on eaten squares).

Let $(v_0, v_1, \ldots, v_n)$ be a journey.

We define a directed graph on $G = \{0, 1, \ldots, n\}$. For each $i \in G$, $i \neq 0$, let j be the least non-negative integer for which $d(v_j, v_i) \leq p$. (Thus $j < i$, since $d(v_{i-1}, v_i) \leq p$.) We connect i to j by a directed edge for each $1 \leq i \leq n$. So the graph G has n directed edges. There is a unique path from n to 0. Let us denote this path by $(a_k, a_{k-1}, \ldots, a_0)$ where k is the number of vertices in this path. Thus $a_k = n$ and $a_0 = 0$. For each $0 \leq i \leq k$ let

$$u_i = v_{a_i}. \tag{1}$$

We call the sequence $(u_0, u_1, \ldots, u_k)$ the *reduced journey* of the journey $(v_0, v_1, \ldots, v_n)$, provided that $v_n \neq v_0$. If $v_n = v_0$ then the reduced journey is defined to be (u_0) and k is defined to be 0. Notice that $u_0 = (0, 0)$, $u_k = v_n$, and $d(u_i, u_{i+1}) \leq p$ for each $0 \leq i < k$. Moreover, for each $1 \leq i \leq k$, if j is the smallest index for which $d(v_j, u_i) \leq p$, then $u_{i-1} = v_j$. Hence for each two distinct i and j we have $u_i \neq u_j$.

Now we shall define the Nice Devil's winning strategy Ψ. The Nice Devil on his first turn (while the Angel is still at $(0, 0)$) eats the same square the Devil would eat on his first turn (which is $\Phi(((0, 0)))$ to be precise), except if the Devil would eat $(0, 0)$, in this case the Nice Devil eats nothing. Given an arbitrary journey $(v_0, v_1, \ldots, v_n)$, let us denote its reduced journey by $(u_0, u_1, \ldots, u_k)$, let $z = \Phi((u_0, u_1, \ldots, u_k))$, and then define

$$\Psi((v_0, v_1, \ldots, v_n)) = \begin{cases} z & \text{if } d(z, v_l) > p \text{ for each } 0 \leq l < n \text{ and } z \neq (0,0), \\ \text{nothing} & \text{otherwise.} \end{cases} \tag{2}$$

Thus the Nice Devil's strategy (Ψ) is to eat the square the Devil (Φ) would eat *for the reduced journey*, but only if he is allowed to eat this square, otherwise he eats nothing; since he is nice.

We claim that Ψ is a winning strategy for the Nice Devil; that is, the Angel cannot leave the domain $B(N)$ whilst playing against Ψ.

Claim 2.6. *If $(v_0, v_1, \ldots, v_n)$ is an allowed journey of the Angel against Ψ (that is, the Angel is able to make this journey without ever jumping on previously eaten squares), then its reduced journey, $(u_0, \ldots, u_k)$, is an allowed journey against the Devil strategy Φ.*

Proof. Suppose that it is not. Thus, there exist integers s and t such that $0 \leq s < t \leq k$ and $\Phi(u_0, \ldots, u_s) = u_t$. Let s' be the smallest index such that $v_{s'} = u_s$, and let t' be the smallest index such that $v_{t'} = u_t$ (see (1) and conclude that $s' = a_s$ and $t' = a_t$), hence $s' < t'$. Notice that since $t \geq 1$, $u_t \neq (0, 0)$, thus $v_{t'} \neq (0, 0)$.

It is easy to see that the reduced journey of $(v_0, v_1, \ldots, v_{s'})$ is $(u_0, \ldots, u_s)$. Hence $\Psi((v_0, v_1, \ldots, v_{s'}))$ is either $\Phi((u_0, \ldots, u_s)) = u_t = v_{t'}$ or 'nothing'. It cannot be $v_{t'}$ since then the Angel has jumped on the previously eaten square $v_{t'}$ while playing against the Nice Devil. So $\Psi((v_0, v_1, \ldots, v_{s'})) =$ nothing. Then, by (2), (since $v_{t'} \neq (0, 0)$) there exists an $l \in \{0, \ldots, s' - 1\}$ such that

$$d(v_{t'}, v_l) \leq p. \tag{3}$$

We claim that this contradicts the fact that $(u_0, \ldots, u_k)$ is the reduced journey of $(v_0, v_1, \ldots, v_n)$. Indeed, in the graph G, the unique path from n to 0 at first crosses t' and then crosses s'. But (3) implies that all the vertices which follow t' in the path are among $\{0, \ldots, s' - 1\}$. This finishes the proof of the claim. ☐

So, by Claim 2.6, if the Angel is able to make the journey $(v_0, v_1, \ldots, v_n)$ against the Nice Devil strategy Ψ without jumping on previously eaten squares, then $(u_0, \ldots, u_k)$ is a Devil-allowed journey against Φ. Hence $u_k \in B(N)$. Since $v_n = u_k$, we also have $v_n \in B(N)$, thus the Nice Devil can entrap the Angel in the domain $B(N)$. ☐

From Claim 2.1 and Theorem 2.4 we immediately get the following.

Theorem 2.7. *If the Devil can catch the Angel of power p, then there is an N such that the Nice Devil can entrap the Angel of power p in the domain $B(N)$.*

3 Running forward with eyes closed

In this section we shall show that the Angel of sufficiently large power can defeat the Nice Devil.

Actually, our Angel will not jump from one square to another but run along neighbouring (uneaten) squares. That is, if the Angel's present location is the square v_0, he may run along the squares $v_0, v_1, v_2, \ldots, v_n$ *in one move*, provided that for all $0 \leq i \leq n$ we have $d(v_0, v_i) \leq p$, and the squares v_i and v_{i+1} are neighbouring (have one side in common) for each $0 \leq i < n$. For technical reasons, let also $n \leq 2p$ (but any finite bound $\geq 2p$ would do). Of course, at the next step the Angel's starting position is v_n. Obviously the Angel of power p can imitate this *Runner*.

Accordingly, we ask less from the Nice Devil. For our purposes, it is enough if he does not eat squares on which the Runner has already stayed or has run through.

We consider the board as a labyrinth: the eaten squares are blocks with high walls which the Runner cannot cross. Without loss of generality we may ask the Nice Devil to eat up all the squares $\{(x, y) : x < 0\}$ before the game starts. This cannot harm him at all.

At any given time we imagine the Runner to be in a square *and* to be facing in a particular direction. (We will adopt some slightly informal terminology such as 'the Runner's left hand', but we hope that this will be much clearer to the reader than some formal version of it.) Before his first move, the Runner is heading north, puts out his left arm and touches the wall (there is a wall there since the square $(-1, 0)$ is already eaten). He may even close his eyes: he needs only his left arm to find his way. In each of his moves, the Runner runs as much as he can (at most p distance away from where he started) while always touching a wall with his left hand on his left side. So the Runner also turns left or right if he needs to. Since the squares $\{(x, y) : x < 0\}$ are eaten before the game starts, the Runner will always have some wall on his left side. (The Runner may use the same square more than once even in the same move!)

Imagine that the Runner also has some green paint in his left hand and he paints a line along the walls he touches. We have mentioned that the Runner runs as much as he can in each of his moves, but at most p distance away from where he started, and his route cannot be longer than $2p$ squares (see the beginning of this section). To be precise we have to give one more restriction: the Runner is allowed to paint at most, say, $2p$ walls in each of his moves (counted with multiplicity). In that we mean he has to stop running after painting $2p$ times. (For example, if the Runner is on a square which is surrounded by 4 walls (eaten squares), he will turn $2p$ times $90°$ to the right in each of his moves—however, this particular situation will never occur.)

Notice that if we see a painted wall we know which way the Runner went, since the walls are always on his left side. Hence the green line has a natural direction.

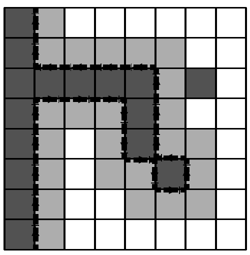

Figure 1: Runner's path

For the sake of simplicity let us call the squares which the Runner has (already) used blue squares. See Figure 1 for an example on how the Runner moves. Dark grey squares are the squares the Nice Devil has eaten, light grey squares are the 'blue' squares the Runner used, and the black dashed line is the 'green' line.

Claim 3.1. *If the Runner uses the strategy defined above, then the following are true:*

1. *At each step, the green (directed) line consists of directed segments of length one. For*

each directed green segment, there is an eaten square on its left side, and there is a blue (uneaten) square on its right side.

2. *Even if the Runner paints the same wall again, he always paints it in the same direction.*
3. *If there is a wall which is painted twice, then the first wall which the Runner has painted twice is the first wall the Runner has ever painted. Therefore, in this case, the green line forms a circle.*

Proof. The first is trivial since the Nice Devil does not eat blue squares. The second comes from the fact that the Runner paints only on his left side.

To prove the third statement, consider the first occasion when the Runner paints a previously painted wall (edge), and let us denote this edge by e. Obviously, if the edge e is the first edge ever painted, then the Runner has painted a 'circle' till painting e for the second time, and he will run around this circle endlessly (actually, on its *inner* side, clockwise).

Suppose that e is not the first edge ever painted. Then the Runner has painted some edge f just before painting e for the first time. We have three significantly different possibilities for the position of these edges (see Figure 2).

If f and e are on the same line, then the two squares on the right of these edges are already blue before the Runner could paint e for the second time, and the two squares on the left are already eaten. Thus the Runner cannot reach e from the right nor from the left, and obviously cannot reach it from above by the second part of this Claim. Thus he had to paint f twice before painting e twice, which contradicts the choice of e. The other two possibilities can be handled the same way as this one, see Figure 2. □

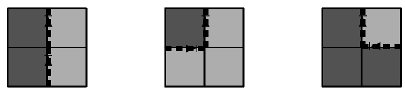

Figure 2: Three possibilities for the position of e and f

It is easy to check that the Runner paints at least p walls in each of his steps (we count this with multiplicity if necessary).

Proposition 3.2. *If p is sufficiently large, then the Runner never goes back to the origin. Moreover, the Runner travels arbitrarily far from the origin, whatever the Nice Devil does.*

The proof is based on the fact that if p is sufficiently large then "the Runner runs faster than the Nice Devil can build walls".

Proof. Notice that if the Runner does not go back to the origin, then by the third part of Claim 3.1, every wall is painted at most once.

Suppose that $p \geq 11$. It is easy to see that whatever the Nice Devil does in his first move, after the first turn of the Runner he is at least p high to the north. Also, after the second turn of the Runner, he is at least $2p$ high to the north.

We shall prove by induction on t that the Runner does not go back to the origin in t steps. For $t = 2$ we are done. Suppose that the Runner didn't go back to the origin in t steps. After t steps, the Nice Devil has eaten up at most t squares, which altogether have at most $4t$ walls. The Runner of power p paints at least tp walls during this time, and these are different walls by our assumption. Thus, the Runner has painted at least $tp - 4t$ walls on the pre-eaten squares (along the y axis). Along the pre-eaten squares he can only go north but not south. Thus, once during this time he was at least $tp - 4t$ high up to the north. He could not go down to the south more than t walls, so after t steps his north coordinate is at least

$$tp - 4t - t = t(p - 5). \tag{4}$$

Since $t \geq 2$ and $p \geq 11$, this is at least $2(p - 5) \geq p + 1$. Thus, the Runner will not go back to the origin in $t + 1$ steps: his north coordinate will be at least one. Hence, by induction, we also get that for each t, after t steps the Runner is at least $t(p - 5)$ high to the north. Thus the Runner travels arbitrarily far from the origin. □

Proposition 3.2 clearly implies that the Angel of sufficiently large power can defeat the Nice Devil. Thus from Theorem 2.7 we obtain the following.

Theorem 3.3. *The Angel of sufficiently large power can defeat the Devil.*

4 The Angel of power 2

To show that the Angel of power 2 can also win, we will slightly modify the way he runs (as compared to the Runner of the previous section) and we will apply a careful calculation. The proof is based on the simple fact that, although a square has four sides, if squares are built next to each other to form a long wall, only two sides per square count. Hence the Angel of power 2 can run around the eaten squares with the same speed as the Devil builds them. (Thus our estimates have to be rather strict and precise—however we will never use that the Angel can also jump over a square.)

We shall consider families of finitely many squares of $\mathbb{Z}^2$. Here we make it clear that the square $(x, y) \in \mathbb{Z}^2$ corresponds to the square $[x, x + 1] \times [y, y + 1]$ in the plane. We call two squares *adjacent* if they have one side in common.

Definition 4.1. A set of squares S (of $\mathbb{Z}^2$) is called *connected* if for each two distinct $a, b \in S$ there exists a path in S, $a = c_0, c_1, \ldots, c_n = b$ such that c_i and c_{i+1} are adjacent for each $0 \leq i < n$.

Claim 4.2. *Let S be a set of n squares of $\mathbb{Z}^2$ ($n \geq 1$). If S is connected, then the boundary of $\cup S$ consists of at most $2n + 2$ sides (unit segments).*

Proof. The proof is trivial by induction. For $n = 1$ the statement holds. Gluing an other square to a connected set of squares deletes (at least) one segment and gives at most three more. It remains only to check that every connected set of $n + 1$ squares can be obtained by gluing one square to a connected set of n squares. □

In this section we fix the Angel's power to be 2. The Angel will act something like the Runner in the previous section, but here he may also make *diagonal* moves (like from $(5, 5)$ to $(4, 6)$). Here the Angel imagines that the base of the blocks the Nice Devil builds are not like squares but like octagons (see Figure 3). Thus, the squares which have one point in common but not one side, become far from each other; the Angel can squeeze through himself between them. Indeed he will, since his left hand will follow the walls. See Figure 3 for an example on how the Runner of the previous section and how the Angel of power 2 moves.

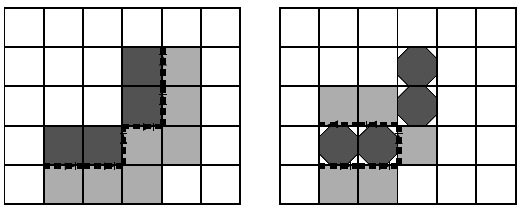

Figure 3: The difference between the Runner and the Angel of power 2

To make it more precise, if the Angel is to move and his location is the square v_0, then he will follow a path $v_0, v_1, \ldots, v_n$ where $d(v_0, v_i) \leq 2$ for all $0 \leq i \leq n$, $d(v_i, v_{i+1}) \leq 1$ for each $0 \leq i < n$, and we also suppose for technical reasons that n is at most 4 (but any larger finite bound would do as well). The Angel's location will be v_n at the end of his move of course. It is enough for our purposes if the Nice Devil does not eat the squares the Angel has used.

Just as in the previous section, the Angel will try to run as much as he can in each of his moves, while his left hand is always touching and painting a wall. (For technical reasons we do not allow the Angel to paint more than 4 walls per turn.) As it can be seen in Figure 3, oddly the paint is on the side of the squares and not on the octagons, this is what we need for our calculations.

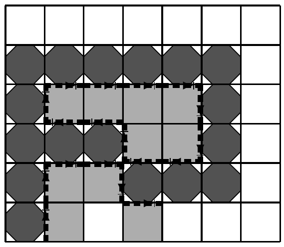

Figure 4: Angel's green line

It is easy to check that the Angel paints at least 2 walls per turn. Thus, in t steps he paints at least $2t$ walls (counted with multiplicity if necessary).

Since the square (x, y) corresponds to the square $[x, x+1] \times [y, y+1]$ in the plane, the Angel's green line starts from $(0, 0) \in \mathbb{R}^2$.

It is easy to check that Claim 3.1 holds for this Angel and his green line as well (only the proof of the third part is different, see also Figure 4). Hence until the Angel's green line reaches the x axis again, the Angel cannot paint twice that wall which he paints in the very beginning of the game, and thus he paints every wall at most once. We will use this fact later.

We also suppose that the Nice Devil has eaten all the squares $\{(x, y) : x < 0\}$ before the game starts.

Proposition 4.3. *The Angel's green line will never reach the x axis again.*

Proof. Consider the first occasion the green line touched the x axis. Suppose that this happened in the Angel's t^{th} turn. Hence the Nice Devil has eaten already (at most) t squares, and the Angel has painted already at least $2(t-1)+1$ walls: $2(t-1)$ in his first $t-1$ moves, and he had to paint at least one more in this turn to reach the x axis. It is easy to conclude from the preceding that these are different walls. Let d be the number of squares the Nice Devil ate in the region $\{(x, y) : x \geq 0, y \geq 0\}$. Let a be the number of different walls the Angel painted till his green line reached the x axis. Thus

$$d \leq t \quad \text{and} \quad a \geq 2t - 1. \tag{5}$$

Let us interrupt the game at this point. Let us delete everything from the lower half of the board: that is, delete all the eaten and the pre-eaten squares of $\{(x, y) : y < 0\}$. (There is no green line on this half plane.) Let us also delete all pre-eaten squares except for those $\{(-1, y) : 0 \leq y \leq N-1\}$ for some very large integer N. Let $N - 100$ be larger than the y coordinate of the uppermost eaten square which the Nice Devil ate during the game, and also larger than the y coordinate of the uppermost point of the green line. Hence there remained exactly $N + d$ eaten squares on the board.

Now let us ask the Angel to continue his journey and his line-painting as he did before. But we will not allow the Nice Devil to eat any more squares. Sooner or later the Angel will return to his starting position in the square $(0, 0)$, and the green line will return to the point $(0, 0)$. Hence the green line will form a circle. Let us denote its length by l (this is also the total number of walls painted).

Claim 4.4.

$$l \geq a + 2N + 5$$

Proof. The Angel painted a different walls during the regular game till the green line reached the x axis, say at the point $(x_0, 0) \in \mathbb{R}^2$. Thus x_0 is an integer and $x_0 \geq 1$. Then the green line surely covered $[(x_0, 0), (x_0+1, 0)]$. Till the green line reaches the point $(0, N)$, it has to go to the north at least N times and to the west at least $x_0 + 1 \geq 2$ times. Then the green line covers $[(0, N), (-1, N)]$ and $[(-1, N), (-1, 0)]$ (which is the left side of the pre-eaten squares $\{(-1, y) : 0 \leq y \leq N-1\}$). Finally the green line covers $[(-1, 0), (0, 0)]$. Therefore $l \geq a + 1 + N + 2 + 1 + N + 1 = a + 2N + 5$. □

It is easy to check that the squares which have at least one side painted, form a connected family. (This is not true for the Runner but true for our more sophisticated Angel of power 2.) This connected family has at least l sides on its boundary (maybe much more). Thus, the number of squares in this family is at least $\frac{l-2}{2}$ by Claim 4.2. On the other hand, there is at most $N+d$ squares in this family, thus

$$N + d \geq \frac{l-2}{2}.$$

From Claim 4.4 we obtain that

$$N + d \geq \frac{a + 2N + 5 - 2}{2} = \frac{a+3}{2} + N.$$

By (5),

$$N + t \geq \frac{2t - 1 + 3}{2} + N = N + t + 1,$$

which is a contradiction. Therefore the green line cannot meet the x axis again during the game. □

Theorem 4.5. *The Angel of power* 2 *can defeat the Nice Devil.*

Proof. We have given a strategy for the Angel of power 2 such that the imaginary green line never reaches the x axis again, and thus the Angel never paints the same wall again (Proposition 4.3). Since in a bounded domain there are only routes of finite length where the green line could go, obviously the green line will leave every bounded domain—hence the Angel also will leave every bounded domain, however the Nice Devil plays. □

Thus by Theorem 2.7 we immediately obtain the following.

Theorem 4.6. *The Angel of power* 2 *can defeat the Devil.*

5 Closing remarks

In Sections 3 and 4 we gave an explicit strategy for the Angel to escape from the Nice Devil. Hence by Section 2, the Angel has a strategy to escape from the Devil. But what does this strategy look like?

It is not trivial to deduce it from Section 2, since there we showed only how the Devil's strategy can be transformed into a strategy of the Nice Devil.

However, it is possible to give a transformation of the strategy against the Nice Devil into a winning strategy against the Devil; a complicated version of the 'short-circuit' method employed in [3, Theorem 8.1] can be used.

We have shown that the Angel can escape from the Devil if and only if his power is at least 2. To be precise, this is true if we restrict ourselves to integer powers. M. Kutz and A. Pór in [7] generalised the game to fractional powers. They introduced the King whose *speed* (power) can be any positive real number. For example, the King of speed 2 (the 2-King) can make two consecutive king's moves on *uneaten* squares on his turn. (Notice the difference between the Angel and the King of the same power.) Here we consider "remaining on the same square" as a valid king move. Fractional speeds are handled by alternating Devil and King moves in a suitable order with densities that correspond to the speed of the King. See [7] for precise definitions and details.

It is shown in [7] that the Devil can build up a 'self-similar' trap to catch the King of speed less than 2.

Observe that in Section 4 we actually showed that the *King of speed* 2 can defeat the Nice Devil (we have to carefully consider what we mean by 'Nice' in this case). It is possible (with some work) to modify Section 2 so that it applies to the King of speed 2 instead of the Angel of power 2. This implies that the King of speed 2 can defeat the Devil, and the following holds.

Theorem 5.1. *Using the terminology of [7], the King of speed p (the p-King) can escape from the Devil if and only if $p \geq 2$.*

M. Kutz and A. Pór generalised the game to fractional powers (as mentioned above) while keeping the discrete lattice $\mathbb{Z}^2$. However, there are many natural generalisations of the game in the plane $\mathbb{R}^2$. We give one example. Let the King be a point in the plane, who is allowed to run along a curve of length at most p in each of his steps. In each of his moves the Devil may draw (eat) a finite union of closed segments of total length one, with the restriction that he cannot eat inside the 1-neighbourhood of the actual position of the King. The King loses if he crosses any of the segments of the Devil, or if the Devil is able to *encircle* the King in a bounded domain.

It is easy to give the proof of [7] in this setting, to show that the Devil can trap the King of speed smaller than 2. On the other hand, the author strongly believes that the King of speed larger than 2 is able to escape forever from the Devil, since he can run faster than the Devil can build walls (segments) as every segment has only two sides.

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Department of Analysis
Eötvös Loránd University
Pázmány Péter sétány 1/c, 1117 Budapest, Hungary
E-mail address: amathe@cs.elte.hu
http://amathe.web.elte.hu